Differentiation as a (co)homological result

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After the great era of classical geometry, new algebraic tools were born to solve geometrical problems. Theses tend to generalize some results on manifolds by the study of their smooth functions commutative algebras. This algebraic setting leads naturally to the introduction of non-commutative geometry while consider bigger classes of algebra which are non-necessarily commutative. Deep non-commutative machines have been developed to understand theses algebras, as **K**-theory, algebraic (co)homologies, characteristic classes, etc...

The main purpose of this talk is to understand the notion of differentiation on a manifold as a (co)homological result on its functions algebra. This result is due to Hochschild-Kostant-Rosenberg known as **HKR**-theorem. The notion of differentiation we want to extend is encapsulated in the data of vector fields and differential forms. They will correspond to derivations and differential forms over an algebra.

$$\mathfrak{Top} \overset{\mathcal{C}^\infty(-)}{\dashrightarrow} \mathfrak{Alg}$$

Manifolds	M	A	Algebras
Vector fields	$\Gamma(TM)$	$Der(A)$	Derivations
Differential forms	$\Gamma(T^\star M)$	$\Omega^1 A$	Algebraic differential forms

Through the smooth functions functor $\mathcal{C}^\infty(-)$, we will present purely algebraic definitions of vector fields and differential forms. It will turn out this algebraic point of view encapsulates deep structures and can be recovered via (co)homological tools.

1 Algebraic differential forms and derivations

We know that any smooth vector field X on a manifold M over $k = \mathbb{R}, \mathbb{C}$ corresponds a linear map

$$\begin{array}{rccc} X: & \mathcal{C}^\infty(M) & \longrightarrow & \mathcal{C}^\infty(M) \\ & f & \longmapsto & X(f) : (x \mapsto \partial_{\overrightarrow{X(x)}} f) \end{array},$$

where $\partial_{\{-\}}$ is the directional derivative, and verifies for any two smooth maps $f, g \in \mathcal{C}^\infty(M)$:

$$X(fg) = fX(g) + X(f)g.$$

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In order to mimic the geometric situation from an algebraic point of view, we fix A to be an algebra over $k = \mathbb{R}, \mathbb{C}$ and we define the following.

DEFINITION 1.1 *Let M be a bimodule over A. A **derivation** D over M is a (k-)linear map $D : A \to M$ respecting the Leibniz rule :*

$$D(ab) = aD(b) + D(a)b. \tag{1}$$

We will write $\boldsymbol{Der(A, M)}$ for the space of derivations over M.

Naturally, any vector field $X \in \mathfrak{X}(M)$ on a manifold M is a derivation over its algebra of smooth functions and the contrary is also true, which is

$$\Gamma(TM) = \mathfrak{X}(M) \xrightarrow{\sim} Der(\mathcal{C}^\infty(M))) \tag{2}$$

The space of geometric differential forms $\Omega^1 M := \Gamma(T^\star M)$ corresponds to the $\mathcal{C}^\infty(M)$-module-dual (which we will denote $(-)^\vee$) of the space $\mathfrak{X}(M)$. Indeed, as $\Gamma(M, E)^\vee = \Gamma(M, E^\star)$ for any finite dimensional fiber bundle E over M, we naturally write :

$$\mathfrak{X}(M)^\vee \simeq \Omega^1 M \text{ and } \mathfrak{X}(M) \simeq (\Omega^1 M)^\vee.$$

To imitate the situation, we want the algebraic differential forms $\Omega^1 A$ to be the A-module-dual of $Der(A)$, which is :

$$Der(A) \simeq \mathrm{Hom}_A(\Omega^1 A, A),$$

i.e. we want to associate in a one-to-one correspondence any derivation to a A-linear map $\Omega^1 A \to A$. To obtain this we will exhibit what we call a *universal derivation* on A.

DEFINITION 1.2 *A **universal derivation** over A is a derivation $d : A \to M$ such that for any other derivation $D : A \to N$, it exists a A-bilinear map $\psi_D : M \to N$ making commute the the following diagram :*

$$\begin{tikzcd} A \arrow[r, "d"] \arrow[dr, "D"'] & M \arrow[d, dotted, "\psi_D"] \\ & N \end{tikzcd}$$

Remark(s) Any universal derivation has the same space of arrival since for any other universal derivation $d' : A \to M'$, we obtain maps $\psi'_d : M \to M'$ and $\psi_d : M' \to M$ which are inverses of each other :

$$\begin{tikzcd} A \arrow[r, "d"] \arrow[dr, "d'"'] & M \arrow[d, dotted, shift right, "\psi'_d"'] \\ & M' \arrow[u, dotted, shift right, "\psi_d"'] \end{tikzcd}$$

DEFINITION 1.3 *The space of **algebraic differential forms** $\Omega^1 A$ over A is the space of arrival of a universal derivation over A.*

Now, by definition of algebraic differential forms, the map $D \mapsto \psi_D$ defines the isomorphism :

$$Der(A) \simeq \mathrm{Hom}_A(\Omega^1 A, A). \tag{3}$$

The space $\Omega^1 A$ turns out to be the A-module-dual of $Der(A)$, as expected. We say that $\Omega^1 A$ represents the functor $Der(-)$.

As the arrival space of the universal derivation d, the space of algebraic differential forms is naturally generated by elements of the form $a_0 d(a_1)a_2$ with $a_0, a_1, a_2 \in A$. But because of the Leibniz rule (1), $a_0 d(a_1)a_2 = a_0 d(a_1 a_2) - a_0 a_1 da_2$ and then :

$$\Omega^1 A = \langle a_0 da_1 \mid a_0, a_1 \in A\rangle. \tag{4}$$

We may wonder what does the space of universal derivations look alike with the commutative algebra $A = \mathcal{C}^\infty(M)$.

Lemma 1.1 *A universal derivation on $\mathcal{C}^\infty(M)$ is given by the following, for any $f \in \mathcal{C}^\infty(M)$:*

$$df : x \in M \mapsto (\vec{v_x} \mapsto \partial_{\vec{v_x}} f) \in T^\star M$$

where $\vec{v_x} \in T_x M$.

Proof. We take a derivation on $\mathcal{C}^\infty(M)$, which is a vector field $X \in \mathfrak{X}(M)$. If we define $\psi_X : \Omega^1\mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ to be the evaluation at X, we show that d is universal :

$$\psi_X(df)(x) = df(x)(\overrightarrow{X(x)}) = \partial_{\overrightarrow{X(x)}} f = X(f)(x).$$

QED.

Since $\Omega^1\mathcal{C}^\infty(M)$ is generated by elements of the form fdg, for $f, g \in \mathcal{C}^\infty(M)$, we obtain as a direct corollary the description :

$$\Gamma(T^\star M) = \Omega^1 M \xrightarrow{\sim} \Omega^1\mathcal{C}^\infty(M), \tag{5}$$

which is analogous to (2). In other words, the algebraic definitions of derivations and differential forms give a generalization of the usual geometrical ones.
What may interests us is to understand how far an algebra is close to be commutative. This data in encapsulated in the (co)homology of Hochschild.

2 Hochschild homology

We still take A to be an (unitary) algebra over $k = \mathbb{R}, \mathbb{C}$.
We define the **Hochschild complex** with coefficients in a A-bimodule M to be :

$$C_\star(A, M) : \ 0 \xleftarrow{b} M \xleftarrow{b} M \otimes A \xleftarrow{b} M \otimes A^{\otimes 2} \xleftarrow{b} \cdots \xleftarrow{b} M \otimes A^{\otimes n} \xleftarrow{b} \cdots$$

where the differential b is given by :

$$\begin{aligned} b(m \otimes a_1 \otimes \cdots \otimes a_n) := &\sum_{i=1}^{n-1} (-1)^i m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\ &+ m a_1 \otimes a_2 \otimes \cdots \otimes a_n + (-1)^n a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1} \end{aligned}$$

In low degrees we have :

$$b(m \otimes a_1) = ma_1 - a_1 m,$$
$$b(m \otimes a_1 \otimes a_2) = ma_1 \otimes a_2 - m \otimes a_1 a_2 + a_2 m \otimes a_1,$$
$$\dots$$

Lemma 2.1 *One may check that $b \circ b = 0$.*

DEFINITION 2.1 *We define the **Hochschild homology** with coefficients in a A-bimodule M to be the data of the vector spaces :*

$$HH_n(A, M) := H_n(C_\star(A, M)).$$

When $M = A$, we use the notation $HH_\star(A) := HH_\star(A, A)$.

For example, let us compute the 0^{th}-degree homology group for $M = A$. It is given by :

$$HH_0(A) = \ker(b : A \to 0)/Im(b : A \otimes A \to A) = A/\langle a_0a_1 - a_1a_0\rangle = A/[A, A].$$

It calculates the degree of commutativity of the algebra. In particular, if A is commutative, $HH_0(A) = A$.

LEMMA 2.2 *If A is commutative, then :*

$$HH_1(A) \simeq \Omega^1 A.$$

Proof. The 1^{st}-degree homology group is given by :

$$HH_1(A) = \ker(b : A \otimes A \to A)/Im(b : A \otimes A \otimes A \to A \otimes A).$$

As A is commutative, $b(a_0 \otimes a_1) = a_0a_1 - a_1a_0 = 0$, so the kernel of $b : A \otimes A \to A$ is exactly $A \otimes A$. We naturally define the map :

$$\varphi : A \otimes A \longrightarrow \Omega^1 A,\ a_0 \otimes a_1 \longmapsto a_0d(a_1).$$

Then the differential $b : A \otimes A \otimes A \to A \otimes A$ given in terms of differential forms is

$$(\varphi \circ b)(a_0 \otimes a_1 \otimes a_2) = a_0a_1d(a_2) - a_0d(a_1a_2) + a_0d(a_1)a_2,$$

because A is supposed commutative. The fact that d is a derivation, tells us that $\varphi \circ b$ is always zero and that the only way for φ to vanish is that we are in the image of b. The map φ descends as an isomorphism in the quotient $\overline{\varphi} : HH_1(A) \to \Omega^1 A$, which is the expected result. QED.

3 Hochschild cohomology

We define the **Hochschild co-complex** with coefficients in a A-bimodule M to be :

$$C^\star(A, M) : \ 0 \xrightarrow{b^\star} M \xrightarrow{b^\star} Hom_A(A, M) \xrightarrow{b^\star} \cdots \xrightarrow{b^\star} Hom_A(A^{\otimes n}, M) \xrightarrow{b^\star} \cdots$$

where the differential $b^\star$ is given by :

$$(b^\star f)(a_1 \otimes \cdots \otimes a_n) = \sum_{i=1}^{n-1} (-1)^{i+1} f(a_1 \otimes \cdots \otimes a_ia_{i+1} \otimes \cdots \otimes a_n) + \\ + a_1 \cdot f(a_2 \otimes \cdots \otimes a_n) + (-1)^n f(a_1 \otimes \cdots \otimes a_{n-1}) \cdot a_n$$

In low degree we have :

$$(b^\star a)(a_1) = aa_1 - a_1a$$

$$(b^\star f)(a_1 \otimes a_2) = -f(a_1a_2) + a_1f(a_2) + f(a_1)a_2$$

$$\cdots$$

Lemma 3.1 *One may check that $b^\star \circ b^\star = 0$.*

Definition 3.1 *We define the **Hochschild cohomology** with coefficients in a A-bimodule M to be the data of the vector spaces :*

$$HH^n(A, M) := H^n(C^\star(A, M)).$$

When $M = A$, we use the notation $HH^\star(A) := HH^\star(A, A)$.

As before let us compute the 0^{th}-degree cohomology group for $M = A$. It is given by :

$$HH^0(A) = \ker(b^\star : A \to Hom_A(A, A))/Im(b^\star : 0 \to A) = \ker(a \mapsto (a_1 \mapsto aa_1 - a_1a)) = Z(A).$$

As in the homology case, it compute the degree of commutativity of A. If A is commutative, then its center is as bigger as possible, which is $HH^0(A) = A$.

Lemma 3.2 *If A is commutative, then*

$$HH^1(A) \simeq Der(A).$$

Proof. The 1^{st}-degree cohomology group is given by :

$$HH^1(A) = \ker(b^\star : Hom_A(A, A) \to Hom_A(A \otimes A, A))/Im(b^\star : A \to Hom_A(A, A)).$$

As A is commutative, $(b^\star a)(a_1) = aa_1 - a_1a = 0$ so we quotient by a trivial space. Now,

$$(b^\star f)(a_1 \otimes a_2) = -f(a_1a_2) + a_1f(a_2) + f(a_1)a_2$$

is zero exactly when f is a derivation (see 1), which gives the expected result. QED.

To conclude, we have shown that algebraic derivation and differential forms could be obtained as (co)homological results. We will show with deeper arguments that we have analogous descriptions in higher degrees.

4 Tor derived functor

Regarding any A-bimodule as a module over the algebra $A \otimes A^{op}$, we are interested in the functor :

$$\begin{array}{ccc} A \otimes A^{op} - \mathfrak{mod} & \longrightarrow & A - \mathfrak{mod} \\ M & \longmapsto & M/[A, M] = M \otimes_{A \otimes A^{op}} A \end{array} \tag{6}$$

With $M = A$, we know that $HH_0(A) = A \otimes_{A \otimes A^{op}} A$.

Lemma 4.1 *For any algebra A, the functor $- \otimes_{A \otimes A^{op}} A$ is left-exact.*

Naturally, as any left-exact functor in a category of modules we want to compute its derived functor. To do so, we need sufficiently what we call a free resolution.

Definition 4.1 *A **free resolution** of a A-bimodule M is a complex $(P_\star, \partial)$ such that all the P_i are free over A and such that the following is exact :*

$$0 \longleftarrow A \xleftarrow{\epsilon} P_0 \xleftarrow{\partial} P_1 \xleftarrow{\partial} \cdots$$

THEOREM 4.2 *For any free resolution of the A-bimodule A, we have :*

$$HH_\star(A) = H_\star((P_\star, \partial) \otimes_{A \otimes A^{op}} A)$$
$$HH^\star(A) = H^\star(Hom_{A \otimes A^{op}}((P_\star, \partial), A))$$

The main example of free resolutions that can compute this (co)homology are the Koszul complexes, that we will present in the following section.

5 Koszul complexes

We still fix A to be an algebra over $k = \mathbb{R}, \mathbb{C}$. Koszul complexes are natural free resolutions for A-modules of the form A/I for a nice enough ideal $I \subset A$.

DEFINITION 5.1 *Let $s : A^r \to A$ be a A-linear map. We define the* ***Koszul complex*** *associated to s as :*

$$K(A,s): \ 0 \longleftarrow A \xleftarrow{\partial_s} A^r \xleftarrow{\partial_s} \bigwedge^2 A^r \xleftarrow{\partial_s} \cdots \xleftarrow{\partial_s} \bigwedge^{r-1} A^r \xleftarrow{\partial_s} \bigwedge^r A^r \longleftarrow 0$$

where the differential is given by :

$$\partial_s(A_1 \wedge \cdots \wedge A_n) := \sum_{i=1}^{n} (-1)^{i+1} s(A_i) \cdot A_1 \wedge \cdots \wedge \widehat{A_i} \wedge \cdots \wedge A_n.$$

A trivial example of Koszul complex is associated to the A-linear map $\cdot x : A \to A$ of multiplication by an element $x \in A$:

$$K(A, \cdot x): \ 0 \longleftarrow A \xleftarrow{\cdot x} A \longleftarrow 0$$

The linear map $s : A^r \to A$ can always be decomposed as a sum of product of elements of A, i.e. it always exists $x_1, \cdots, x_r \in A$ such that :

$$s(a_1, \cdots, a_r) = \sum_{i=1}^{r} a_i x_i.$$

Also, one can check that $K(A,s) = \bigotimes_{i=1}^{q} K(A, \cdot x_i)$, where the tensor product in taken over the chain complexes category.

DEFINITION 5.2 *We say that $(x_1, \cdots, x_r) \in A^r$ is a* ***regular sequence*** *if x_i is not a divisor of zero in $A/(x_1, \cdots, x_{i-1})$, for all i.*

PROPOSITION 5.1 *If $s = (x_1, \cdots, x_r) \in A^r$ is a regular sequence, then $K(A,s)$ defines a free resolution of the algebra $A/(x_1, \cdots, x_r)$.*

Proof. It is clear that every $K(A,s)_n$ is free over A, and that the cokernel of $\partial_s : A^r \to A$ is $A/(x_1, \cdots, x_r)$ in that case. QED.

Example(s) If $X = \sum_{i=0}^{q} f_i \, \partial/\partial x_i$ is a vector field over a q-dimensional manifold M, then the space of smooth functions over the vanishing points of X can be obtained as a quotient of $\mathcal{C}^\infty(M)$ by an ideal :

$$\mathcal{C}^\infty(X^{-1}(0)) \simeq \mathcal{C}^\infty(M)/(f_1, \cdots, f_q).$$

If furthermore the vector field is generated by a regular sequence $(f_1, \cdots, f_q)$, which is that X is of maximal rank, the Koszul complex $K(A, (f_1, \cdots, f_q))$ defines a free resolution of the algebra $\mathcal{C}^\infty(X^{-1}(0))$.

As any A-linear map $s : A^r \to A$ is by definition an element of $(A^r)^\vee$, we also have another complex :

$$K'(A,s):\ 0 \longleftarrow \bigwedge^r (A^r)^\vee \xleftarrow{s\wedge\cdot} \bigwedge^{r-1}(A^r)^\vee \xleftarrow{s\wedge\cdot} \cdots \xleftarrow{s\wedge\cdot} (A^r)^\vee \xleftarrow{1\mapsto s} A \longleftarrow 0$$

But the isomorphism $\bigwedge^\star (A^r)^\vee \simeq \bigwedge^{r-\star} A^r$ sends the differential $s\wedge\cdot$ to ∂_s, which gives the isomorphism of complexes $K'(A,s) \simeq K(A,s)$.

Example(s) As the space of differential forms is the $\mathcal{C}^\infty(M)$-module-dual of $\mathfrak{X}(M)$, then for any vector field $X \in \mathfrak{X}(M)$, the two following complexes compute the same homology groups :

$$0 \longleftarrow \bigwedge^r \mathfrak{X}(M) \xleftarrow{X\wedge\cdot} \bigwedge^{r-1}\mathfrak{X}(M) \xleftarrow{X\wedge\cdot} \cdots \xleftarrow{X\wedge\cdot} \mathfrak{X}(M) \xleftarrow{1\mapsto X} \mathcal{C}^\infty(M) \longleftarrow 0$$

$$0 \longleftarrow \mathcal{C}^\infty(M) \xleftarrow{\iota_X} \Omega^1 M \xleftarrow{\iota_X} \cdots \xleftarrow{\iota_X} \Omega^{r-1} M \xleftarrow{\iota_X} \Omega^r M \longleftarrow 0$$

Proposition 5.2 *For every A-linear map $s : A^r \to A$:*

$$K(A,s)^\vee \simeq K(A,s)[r], \tag{7}$$

where $(-)[r]$ means that we shifted the complex by r degrees.

Proof. Via the description $(\bigwedge^\star A^r)^\vee \simeq \bigwedge^\star (A^r)^\vee$, the differential $\partial_s^\vee$ becomes $(-1)^r s\wedge\cdot$, which gives $K(A,s)^\vee \simeq K'(A,s)[r]$. With the remark above, we obtain the expected result. QED.

This property is know as the Koszul self-duality. It is responsible of lot of dualities that we can recover with statements in *equivariant homotopy theory*, *derived categories* and *representation theory*.

Now, to fit Koszul complexes in our study, we need the following statement.

Lemma 5.3 *When a manifold M with commutative algebra $A = \mathcal{C}^\infty(M)$ is a complete intersection, the kernel of the multiplication $m : A\otimes A^{op} \to A$ is generated by a regular sequence $(B_1, \cdots, B_q) \in (A\otimes A^{op})^q$ where $q = \dim(M)$.*

Thanks to this result, we know that in these circumstances, the Koszul complex $\boldsymbol{K} := K(A\otimes A^{op}, (B_1,\cdots,B_q))$ defines a resolution of $A\otimes A^{op}/\ker(m)$, which is naturally isomorphic to A. In other words, when $A = \mathcal{C}^\infty(M)$ is the commutative algebra of a complete intersection, we can compute its Hochschild (co)homology as follows :

$$HH_\star(A) = H_\star(\boldsymbol{K} \otimes_{A\otimes A^{op}} A),$$

$$HH^\star(A) = H^\star(Hom_{A\otimes A^{op}}(\boldsymbol{K}, A)).$$

Theorem 5.4 (*HKR-theorem*) *If $A = \mathcal{C}^\infty(M)$ is the commutative algebra of a complete intersection, then :*

$$HH_n(A) \simeq \Omega^n A \text{ and } HH^n(A) \simeq \bigwedge^n Der(A). \tag{8}$$

Proof. We have to understand the Koszul complex $\boldsymbol{K}$ through the functors $-\otimes_{A\otimes A^{op}} A$ and $Hom_{A\otimes A^{op}}(-, A)$. First of all, the copy of $A = A\otimes A^{op}/\ker(m)$ at the right kills any occurrence of the multiplication map. But in all the terms of the Koszul differential on $\boldsymbol{K}$ appears a certain multiplication by B_i. In other words, the differentials in $\boldsymbol{K}\otimes_{A\otimes A^{op}} A$ and $Hom_{A\otimes A^{op}}(\boldsymbol{K}, A)$ are both zero. So we have :

$$HH_n(A) = \boldsymbol{K}_n \otimes_{A\otimes A^{op}} A \text{ and } HH^n(A) = Hom_{A\otimes A^{op}}(\boldsymbol{K}_n, A).$$

Also, $\boldsymbol{K}$ is determined by a regular sequence of length the dimension of the underlying manifold, which gives the description $\bigwedge^n (A\otimes A^{op})^q \simeq \Omega^n(A\otimes A^{op})$. It follows by straight computations on base changes that :

$$HH_n(A) = \Omega^n(A\otimes A^{op}) \otimes_{A\otimes A^{op}} A \simeq \Omega^n A,$$

$$HH^n(A) = Hom_{A\otimes A^{op}}(\Omega^n(A\otimes A^{op}), A) \simeq Hom_A(\Omega^n A, A) \simeq \bigwedge^n Der(A).$$

QED.

Remark(s) These isomorphisms are really often denoted ε, and are known as *anti-symmetrization map*. These are canonical and define an natural chain complex homotopy from two distinct resolutions of a commutative algebra as a bimodule over itself.

The Hochschild-Kostant-Rosenberg theorem is the key point that justify the study of algebras for the understanding of manifolds. This approach is relevant as the Hochschild (co)homology of a commutative algebra is nothing else than the space of sections over a vector bundle : *some purely topological informations are encapsulated in purely algebraic computations.*